

# Intertwining Operator Realization of anti de Sitter Holography

N. Aizawa<sup>a,1</sup>, V.K. Dobrev<sup>b,2</sup>

<sup>a</sup> *Department of Mathematics and Information Sciences,  
Graduate School of Science, Osaka Prefecture University, Nakamozu Campus,  
Sakai, Osaka 599-8531 Japan*

<sup>b</sup> *Institute of Nuclear Research and Nuclear Energy,  
Bulgarian Academy of Sciences  
72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria*

## Abstract

We give a group-theoretic interpretation of relativistic holography as equivalence between representations of the anti de Sitter algebra describing bulk fields and boundary fields. Our main result is the explicit construction of the boundary-to-bulk operators for arbitrary integer spin in the framework of representation theory. Further we show that these operators and the bulk-to-boundary operators are intertwining operators. In analogy to the de Sitter case, we show that each bulk field has two boundary (shadow) fields with conjugated conformal weights. These fields are related by another intertwining operator given by a two-point function on the boundary.

**Keywords:** anti de Sitter holography, spin representations of  $so(3,2)$ , boundary-to-bulk intertwining operators

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<sup>1</sup>aizawa@mi.s.osakafu-u.ac.jp

<sup>2</sup>dobrev@inrne.bas.bg

# 1 Introduction

For the last fifteen years due to remarkable proposal of [1] the AdS/CFT correspondence is a dominant subject in string theory and conformal field theory. Actually the possible relation of field theory on anti de Sitter space to conformal field theory on boundary Minkowski space-time was studied also before, cf., e.g., [2–7]. The proposal of [1] was further elaborated in [8] and [9]. After these initial papers there was an explosion of related research which continues also currently, cf. e.g., [10]–[53].

Let us remind that the AdS/CFT correspondence has 2 ingredients [1, 8, 9]: 1. the holography principle, which is very old, and means the reconstruction of some objects in the bulk (that may be classical or quantum) from some objects on the boundary; 2. the reconstruction of quantum objects, like 2-point functions on the boundary, from appropriate actions on the bulk.

Our focus is on the first ingredient and we consider explicitly the simplest case of the (3+1)-dimensional bulk with boundary 3D Minkowski space-time. The reason for this is that until now the explicit presentation of the holography principle was realized in the Euclidean case, relying on Wick rotations of the final results, cf. e.g. [9, 28].

Yet it is desirable to show the holography principle by direct construction in Minkowski space-time. This is what we do in the present paper using representation theory only. For this we use a method that is used in the mathematical literature for the construction of discrete series representations of real semisimple Lie groups [54, 55], and which method was applied in the physics literature first in [56] exactly in the Euclidean AdS/CFT setting, though that term was not used then.<sup>3</sup>

The method utilizes the fact that in the bulk the Casimir operators are not fixed numerically. Thus, when a vector-field realization of the anti de Sitter algebra  $so(3, 2)$  is substituted in the bulk Casimir it turns into a differential operator. In contrast, the boundary Casimir operators are fixed by the quantum numbers of the fields under consideration. Then the bulk/boundary correspondence forces an eigenvalue equation involving the Casimir differential operator. That eigenvalue equation is used to find the two-point Green function in the bulk which is then used to construct the boundary-to-bulk integral operator. This operator maps a boundary field to a bulk field. This is our main result. We stress that in our construction the bulk and boundary fields have arbitrary integer spin. This is in sharp contrast to preceding results in the literature which considered spin zero and some very low spin values (in the Euclidean case the intertwiners with arbitrary integer spin have been discussed in [36]).

What is also important in our approach is that we show that this operator is an intertwining operator, namely, it intertwines the two representations of the anti de Sitter algebra  $so(3, 2)$  acting in the bulk and on the boundary.

This also helps us to establish that each bulk field has actually two bulk-to-boundary

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<sup>3</sup>This method was applied recently also to the case of non-relativistic holography [57].

limits. The two boundary fields have conjugated conformal weights  $\Delta, 3 - \Delta$ , and they are related by a boundary two-point function.<sup>4</sup>

The paper is organized as follows. In Section 2 we give the preliminaries on the anti de Sitter algebra, its elementary representations, the vector-field realizations on the boundary and in the bulk. In Section 3 we consider the eigenvalue problem in the bulk and we construct the two-point function in the bulk. In Section 4 we give the bulk-to-boundary operator and construct the integral boundary-to-bulk operator. In Section 5 we establish the intertwining properties of the boundary-to-bulk and bulk-to-boundary operators. We display also the intertwining relation between the two bulk-to-boundary limits of a bulk field.

## 2 Preliminaries

### 2.1 Lie algebra and group

We need some well-known preliminaries to set up our notation and conventions. The Lie algebra  $\mathcal{G} = so(3, 2)$  may be defined as the set of  $5 \times 5$  matrices  $X$  which fulfil the relation:<sup>5</sup>

$${}^tX\eta + \eta X = 0, \quad (2.1)$$

where the metric  $\eta$  is given by

$$\eta = (\eta_{AB}) = \text{diag}(-1, 1, 1, 1, -1), \quad A, B = 0, 1, \dots, 4 \quad (2.2)$$

Then we can choose a basis  $X_{AB} = -X_{BA}$  of  $\mathcal{G}$  satisfying the commutation relations

$$[X_{AB}, X_{CD}] = \eta_{AC}X_{BD} + \eta_{BD}X_{AC} - \eta_{AD}X_{BC} - \eta_{BC}X_{AD}. \quad (2.3)$$

We list the important subalgebras of  $\mathcal{G}$ :

- $\mathcal{K} = so(3) \oplus so(2)$ , generators:  $X_{AB} : (A, B) \in \{1, 2, 3\}, \{0, 4\}$ , maximal compact subalgebra;
- $\mathcal{Q}$ , generators:  $X_{AB} : A \in \{1, 2, 3\}, B \in \{0, 4\}$ , non-compact completion of  $\mathcal{K}$ ;
- $\mathcal{A} = so(1, 1)$ , generator:  $D \doteq X_{34}$ , dilatations;
- $\mathcal{M} = so(2, 1)$ , generators:  $X_{AB} : (A, B) \in \{0, 1, 2\}$ , Lorentz subalgebra;
- $\mathcal{N}$ , generators:  $T_\mu = X_{\mu 3} + X_{\mu 4}$ ,  $\mu = 0, 1, 2$ , translations;

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<sup>4</sup>The conjugated fields are called 'shadow' fields in the physics literature, cf. [58], which terminology was revived in AdS/CFT by Metsaev [33].

<sup>5</sup>For other purposes it may be more convenient to use the other fundamental representation in terms of  $4 \times 4$  matrices as in [59].

- $\tilde{\mathcal{N}}$ , generators:  $C_\mu = X_{\mu 3} - X_{\mu 4}$ ,  $\mu = 0, 1, 2$ , special conformal transformations.
- $\mathcal{H}$ , generators:  $D, X_{12}$ , Cartan subalgebra of  $\mathcal{G}$ ;

Thus, we have several decompositions:

- $\mathcal{G} = \mathcal{K} \oplus \mathcal{Q}$ , Cartan decomposition;
- $\mathcal{G} = \mathcal{K} \oplus \mathcal{A} \oplus \mathcal{N}$ , and  $\mathcal{N} \rightarrow \tilde{\mathcal{N}}$ , Iwasawa decomposition;
- $\mathcal{G} = \mathcal{N} \oplus \mathcal{M} \oplus \mathcal{A} \oplus \tilde{\mathcal{N}}$ , Bruhat decomposition;

The subalgebra  $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \tilde{\mathcal{N}}$  is a maximal parabolic subalgebra of  $\mathcal{G}$ .

Finally, we introduce the corresponding connected Lie groups:  
 $G = SO_0(3, 2)$  with Lie algebra  $\mathcal{G} = so(3, 2)$ ,  $K = SO(3) \times SO(2)$  is the maximal compact subgroup of  $G$ ,  $A = \exp(\mathcal{A}) = SO_0(1, 1)$  is abelian simply connected,  $N = \exp(\mathcal{N}) \cong \tilde{N} = \exp(\tilde{\mathcal{N}})$ , are abelian simply connected subgroups of  $G$  preserved by the action of  $A$ . The group  $M \cong SO_0(2, 1)$  (with Lie algebra  $\mathcal{M}$ ) commutes with  $A$ . The subgroup  $P = MAN$  is a *maximal parabolic subgroup* of  $G$ . Parabolic subgroups are important because the representations induced from them generate all admissible irreducible representations of semisimple groups [60, 61].

## 2.2 Elementary representations

We use the approach of [62] which we adapt in a condensed form here. We work with so-called *elementary representations* (ERs). They are induced from representations of  $P = MAN$ , where we use finite-dimensional representations of spin  $s \in \frac{1}{2}\mathbb{Z}_+$  of  $M$ , (non-unitary) characters of  $A$  represented by the conformal weight  $\Delta$ , and the factor  $N$  is represented trivially. The data  $s, \Delta$  is enough to determine a weight  $\Lambda \in \mathcal{H}^*$ , cf. [62]. Thus, we shall denote the ERs by  $C^\Lambda$ . Sometimes we shall write:  $\Lambda = [s, \Delta]$ . The representation spaces are  $C^\infty$  functions on  $G/P$ , or equivalently, on the locally isomorphic group  $\tilde{N}$  with appropriate asymptotic conditions (which we do not need explicitly, cf. e.g. [63]). We recall that  $\tilde{N}$  is isomorphic to 3D Minkowski space-time  $\mathfrak{M}$  whose elements will be denoted by  $x = (x_0, x_1, x_2)$ , while the corresponding elements of  $\tilde{N}$  will be denoted by  $\tilde{n}_x$ . The Lorentzian inner product in  $\mathfrak{M}$  is defined as usual:

$$\langle x, x' \rangle \doteq x_0 x'_0 - x_1 x'_1 - x_2 x'_2, \quad (2.4)$$

and we use the notation  $\mathbf{x}^2 = \langle x, x \rangle$ .

The representation action is given as follows:

$$(T^\Lambda(g)\varphi)(x) = y^{-\Delta} D^s(m) \varphi(x') \quad (2.5)$$

the various factors being defined from the local Bruhat decomposition  $G \cong_{\text{loc}} \tilde{N}AMN$  :

$$g^{-1} \tilde{n}_x = \tilde{n}_{x'} a^{-1} m^{-1} n^{-1} , \quad (2.6)$$

where  $y \in \mathbb{R}_+$  parametrizes the elements  $a \in A$ ,  $m \in M$ ,  $D^s(m)$  denotes the representation action of  $M$ ,  $n \in N$ .

In the above general definition  $\varphi(x)$  are considered as elements of the finite-dimensional representation space  $V^s$  in which act the operators  $D^s(m)$ . Following [56,62] we use scalar functions over an extended space  $\mathfrak{M} \times \mathfrak{M}_0$ , where  $\mathfrak{M}_0$  is a cone parametrized by the variable  $\zeta = (\zeta_0, \zeta_1, \zeta_2)$  subject to the condition:

$$\zeta^2 = \langle \zeta, \zeta \rangle = \zeta_0^2 - \zeta_1^2 - \zeta_2^2 = 0.$$

The internal variable  $\zeta$  will carry the representation  $D^s$ .

The functions on the extended space will be denoted as  $\varphi(x, \zeta)$ . On these functions the infinitesimal action of our representations looks as follows:

$$\begin{aligned} T_\mu &= \partial_\mu, \quad \partial_\mu \doteq \frac{\partial}{\partial x_\mu}, \quad \mu = 0, 1, 2 \\ D &= - \sum_{\mu=0}^2 x_\mu \partial_\mu - \Delta, \\ X_{01} &= x_0 \partial_1 + x_1 \partial_0 + \mathfrak{s}_{01}, \quad X_{02} = x_0 \partial_2 + x_2 \partial_0 + \mathfrak{s}_{02}, \\ X_{12} &= -x_1 \partial_2 + x_2 \partial_1 + \mathfrak{s}_{12}, \\ C_0 &= 2x_0 D + \mathbf{x}^2 \partial_0 - 2(x_1 \mathfrak{s}_{01} + x_2 \mathfrak{s}_{02}), \\ C_1 &= -2x_1 D + \mathbf{x}^2 \partial_1 + 2(x_0 \mathfrak{s}_{01} - x_2 \mathfrak{s}_{12}), \\ C_2 &= -2x_2 D + \mathbf{x}^2 \partial_2 + 2(x_0 \mathfrak{s}_{02} + x_1 \mathfrak{s}_{12}). \end{aligned} \quad (2.7)$$

where

$$\mathfrak{s}_{01} = \zeta_0 \frac{\partial}{\partial \zeta_1} + \zeta_1 \frac{\partial}{\partial \zeta_0}, \quad \mathfrak{s}_{02} = \zeta_0 \frac{\partial}{\partial \zeta_2} + \zeta_2 \frac{\partial}{\partial \zeta_0}, \quad \mathfrak{s}_{12} = -\zeta_1 \frac{\partial}{\partial \zeta_2} + \zeta_2 \frac{\partial}{\partial \zeta_1}, \quad (2.8)$$

and they satisfy the  $\mathcal{M} = so(1, 2)$  commutation relations

$$[\mathfrak{s}_{01}, \mathfrak{s}_{02}] = -\mathfrak{s}_{12}, \quad [\mathfrak{s}_{02}, \mathfrak{s}_{12}] = \mathfrak{s}_{01}, \quad [\mathfrak{s}_{12}, \mathfrak{s}_{01}] = \mathfrak{s}_{02} . \quad (2.9)$$

The Casimir of  $\mathcal{G}$  is given by:

$$\mathcal{C} = \frac{1}{2} X_{AB} X^{AB} = -X_{01}^2 - X_{02}^2 + X_{12}^2 - D^2 - 3D - C_0 T_0 + C_1 T_1 + C_2 T_2 \quad (2.10)$$

and it is constant on our representation  $C^\Lambda$  :

$$\mathcal{C} \varphi = -(\Delta(\Delta - 3) + s(s + 1)) \varphi = \lambda(s, \Delta) \varphi . \quad (2.11)$$

Note that the constant  $\lambda(s, \Delta)$  has the same value if we replace  $\Delta$  by  $3 - \Delta$ . This means that the two boundary (shadow) fields with conformal weights  $\Delta$  and  $3 - \Delta$  are related, or in mathematical language, that the corresponding representations are (partially) equivalent.<sup>6</sup>

## 2.3 Bulk representations

It is well known that the group  $SO(3, 2)$  is called also anti de Sitter group, as it is the group of isometry of 4D anti de Sitter space:

$$\xi^A \xi^B \eta_{AB} = 1. \quad (2.12)$$

There are several ways to parametrize anti de Sitter space. We shall utilize the same local Bruhat decomposition that we used in the previous subsection. Thus, we use the local coordinates on the factor-space  $G/MN \cong_{\text{loc}} \tilde{N}A$ , i.e., the coordinates  $(x, y) = (x_0, x_1, x_2, y)$ ,  $y \in \mathbb{R}_+$ . In this setting anti de Sitter space is called *bulk* space, while 3D Minkowski space-time is called *boundary* space, as it is identified with the bulk boundary value  $y = 0$ . The functions on the bulk extended with the cone  $\mathfrak{M}_0$  will be denoted by  $\phi(x, y, \zeta)$ .

As we explained in the Introduction we first concentrate on the holography principle, or boundary-to-bulk correspondence, which means to have an operator which maps a boundary field  $\varphi$  to a bulk field  $\phi$ . This map must be invariant w.r.t. the Lie algebra  $so(3, 2)$ . In particular, this means that the Casimir must have the same values in the boundary and bulk representations. The Casimir on the boundary representation  $C^\Lambda$  is a constant  $\lambda(s, \Delta)$  given in (2.11). Clearly, the (partially) equivalent bulk representation  $\hat{C}^\Lambda$  will consist only of functions on which the Casimir has the same value.

To give more precisely the  $\hat{C}^\Lambda$  we first give a vector-field realization of  $so(3, 2)$  on the bulk functions  $\phi(x, y, \zeta)$ :

$$\begin{aligned} T_\mu &= \partial_\mu, \quad \mu = 0, 1, 2 \\ D &= - \sum_{\mu=0}^2 x_\mu \partial_\mu - y \partial_y, \\ X_{01} &= x_0 \partial_1 + x_1 \partial_0 + \mathfrak{s}_{01}, \quad X_{02} = x_0 \partial_2 + x_2 \partial_0 + \mathfrak{s}_{02}, \\ X_{12} &= -x_1 \partial_2 + x_2 \partial_1 + \mathfrak{s}_{12}, \\ C_0 &= 2x_0 D + (\mathbf{x}^2 + y^2) \partial_0 + 2(y \mathfrak{s}_{12} - x_1 \mathfrak{s}_{01} - x_2 \mathfrak{s}_{02}), \\ C_1 &= -2x_1 D + (\mathbf{x}^2 + y^2) \partial_1 + 2(y \mathfrak{s}_{02} + x_0 \mathfrak{s}_{01} - x_2 \mathfrak{s}_{12}), \\ C_2 &= -2x_2 D + (\mathbf{x}^2 + y^2) \partial_2 + 2(-y \mathfrak{s}_{01} + x_0 \mathfrak{s}_{02} + x_1 \mathfrak{s}_{12}), \end{aligned} \quad (2.13)$$

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<sup>6</sup>We remind that two representations are called *partially equivalent* if there exists an intertwining operator between the two representations. The representations are called *equivalent* if the intertwining operator is onto and invertible.

One may verify by straightforward but lengthy computation that (2.13) satisfies (2.3).

Note that the realization of  $so(3, 2)$  on the boundary given in (2.7) may be obtained from (2.13) by replacing  $y\partial_y \rightarrow \Delta$  and then taking the limit  $y \rightarrow 0$ .

Now we find that the Casimir operator is given in the bulk as follows:

$$\mathcal{C} = \mathcal{C}_B + \mathcal{C}_I - 2y(\mathfrak{s}_{12}\partial_0 - \mathfrak{s}_{02}\partial_1 + \mathfrak{s}_{01}\partial_2), \quad (2.14)$$

$$\mathcal{C}_B = y^2(-\partial_0^2 + \partial_1^2 + \partial_2^2) - y^2\partial_y^2 + 2y\partial_y, \quad (2.15)$$

$$\mathcal{C}_I = (-\mathfrak{s}_{01}^2 - \mathfrak{s}_{02}^2 + \mathfrak{s}_{12}^2) \quad (2.16)$$

where  $\mathcal{C}_I$  is the Casimir operator of  $so(1, 2)$  in terms of the internal variables.

Since the Casimir in the bulk is not constant but a differential operator our representation functions will be found as the Casimir eigenfunctions in the bulk. Thus, we consider the eigenvalue problem of the Casimir operator of  $so(3, 2)$ :

$$\mathcal{C}\phi(x, y, \zeta) = \lambda(s, \Delta)\phi(x, y, \zeta), \quad \phi \in \hat{C}^\Lambda. \quad (2.17)$$

In addition, the elements of  $\hat{C}^\Lambda$  must fulfil the appropriate boundary condition:

$$\phi(x, y, \zeta)|_{y \rightarrow 0} \longrightarrow y^\Delta \phi(x, 0, \zeta), \quad \phi \in \hat{C}^\Lambda. \quad (2.18)$$

Later we shall see that the elements of  $\hat{C}^\Lambda$  fulfil also the boundary condition with  $\Delta \rightarrow 3 - \Delta$  which is natural having in mind the degeneracy of Casimir values for (partially) equivalent representations ( $\Delta \leftrightarrow 3 - \Delta$ ).

Next we mention that the realization (2.13) causes the infinitesimal transformation of the bulk coordinates:

$$\begin{aligned} T_\mu &: x_\mu \rightarrow x_\mu + a, \\ D &: x_\mu \rightarrow (1 - a)x_\mu, \quad y \rightarrow (1 - a)y, \\ X_{0\mu} &: x_0 \rightarrow x_0 + ax_\mu, \quad x_\mu \rightarrow x_\mu + ax_0, \quad \mu = 1, 2 \\ X_{12} &: x_1 \rightarrow x_1 + ax_2, \quad x_2 \rightarrow x_2 - ax_1, \\ C_0 &: x_0 \rightarrow x_0 + a(y^2 - x_0^2 - x_1^2 - x_2^2), \quad x_{1,2} \rightarrow (1 - 2ax_0)x_{1,2}, \\ &\quad y \rightarrow (1 - 2ax_0)y, \\ C_1 &: x_1 \rightarrow x_1 + a(y^2 + x_0^2 + x_1^2 - x_2^2), \quad x_{0,2} \rightarrow (1 + 2ax_1)x_{0,2}, \\ &\quad y \rightarrow (1 + 2ax_1)y, \\ C_2 &: x_2 \rightarrow x_2 + a(y^2 + x_0^2 - x_1^2 + x_2^2), \quad x_{0,1} \rightarrow (1 + 2ax_2)x_{0,1}, \\ &\quad y \rightarrow (1 + 2ax_2)y. \end{aligned}$$

It follows that every  $SO(3, 2)$  invariant of the two points  $(x_\mu, y)$  and  $(x'_\mu, y')$  is a function of

$$u = \frac{4yy'}{(\mathbf{x} - \mathbf{x}')^2 + (y + y')^2}. \quad (2.19)$$

We set  $x'_\mu = 0$ ,  $y' = 1$  and obtain a one-point variable<sup>7</sup> which shall be very useful below:

$$\hat{u} = \frac{4y}{\mathbf{x}^2 + (y+1)^2} . \quad (2.20)$$

### 3 Eigenvalue problem and two-point functions in the bulk

#### 3.1 Eigenvalue problem of Casimir in the bulk

Here we first solve the equation:

$$\mathcal{C}\Psi(x, y, \zeta) = \lambda(s, \Delta)\Psi(x, y, \zeta) \quad (3.1)$$

We are interested in solutions in which the  $\zeta$ -dependence is factored out in the form

$$\Psi = \psi(x, y) Q(x, y, \zeta)^s$$

where  $Q$  is homogeneous in  $\zeta$  of first degree (which is due to the fact that  $\Psi$  is homogeneous in  $\zeta$  of degree  $s \in \mathbb{Z}_+$ ). We assume that  $\psi$  is a  $SO(3, 2)$  invariant, thus it is function only of  $\hat{u}$ :  $\psi(x, y) = \psi(\hat{u})$ . When  $\mathcal{C}_B$  acts on  $\psi$  one may write  $\mathcal{C}_B$  in terms of  $\hat{u}$  only:

$$\mathcal{C}_B = \hat{u}^2(\hat{u} - 1) \frac{d^2}{d\hat{u}^2} + 2\hat{u} \frac{d}{d\hat{u}} . \quad (3.2)$$

Furthermore we require that  $Q$  is an eigenfunction of  $\mathcal{C}_I$ . This will guarantee that the spin part  $Q^s$  is the eigenfunction of  $\mathcal{C}_I$  with the correct spin value. It follows that  $\Psi$  is also an eigenfunction of  $\mathcal{C}_I$ :

$$\mathcal{C}_I \Psi = \lambda_I \Psi = -s(s+1) \Psi . \quad (3.3)$$

With the fixed vector  $(\zeta'_0, \zeta'_1, \zeta'_2)$  in the internal space, we use the following Ansatz for  $Q$

$$Q = \frac{2 I_1 - (\mathbf{x}^2 - (y+1)^2) I_2 - 2(y+1) I_3}{\mathbf{x}^2 + (y+1)^2} , \quad (3.4)$$

$$I_1 = \langle x, \zeta \rangle \langle x, \zeta' \rangle, \quad I_2 = \langle \zeta, \zeta' \rangle, \quad I_3 = \sum_{\mu=0}^2 x_\mu (\zeta \times \zeta')_\mu ,$$

where  $\zeta \times \zeta'$  is the standard vector product. (Note that  $I_1, I_2, I_3$  are the three possible scalars which are homogeneous of first degree in both  $\zeta$  and  $\zeta'$ .) It is easy to verify that

$$\mathcal{C}_I I_k = -2I_k . \quad (3.5)$$

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<sup>7</sup>Sometimes in the literature it is called colloquially 'one-point invariant'.



Thus, we have

$$\mathcal{C}_I Q = -2Q \quad (3.6)$$

and one may verify that  $\mathcal{C}_I Q^s = -s(s+1)Q^s$ . With this form of  $Q$  the eigenvalue problem is reduced to the second order differential equation:

$$\left( (\hat{u} - 1)\hat{u}^2 \frac{d^2}{d\hat{u}^2} + 2\hat{u} \frac{d}{d\hat{u}} - s(s+1)\hat{u} \right) \psi(\hat{u}) = (\lambda - \lambda_I) \psi(\hat{u}) = \Delta(3 - \Delta) \psi(\hat{u}) . \quad (3.7)$$

We sketch the derivation of (3.7). First we observe that

$$(\mathcal{C} - \mathcal{C}_I) \Psi = \Psi \left[ \psi^{-1} \mathcal{C}_B \psi + sQ^{-1}(\mathcal{C} - \mathcal{C}_I)Q + 2s\psi^{-1}Q^{-1}A_1 + s(s-1)Q^{-2}A_2 \right], \quad (3.8)$$

where

$$\begin{aligned} A_1 &= -y^2 (\langle \partial\psi, \partial Q \rangle + (\partial_y \psi)(\partial_y Q)) - y((\partial_0 \psi)\mathfrak{s}_{12}Q - (\partial_1 \psi)\mathfrak{s}_{02}Q + (\partial_2 \psi)\mathfrak{s}_{01}Q), \\ A_2 &= -y^2 (\langle \partial Q, \partial Q \rangle + (\partial_y Q)^2) - 2y((\partial_0 Q)\mathfrak{s}_{12}Q - (\partial_1 Q)\mathfrak{s}_{02}Q + (\partial_2 Q)\mathfrak{s}_{01}Q), \end{aligned}$$

where  $\langle \partial\psi, \partial Q \rangle = (\partial_0 \psi)\partial_0 Q - (\partial_1 \psi)\partial_1 Q - (\partial_2 \psi)\partial_2 Q$ . Straightforward computation shows that

$$(\mathcal{C} - \mathcal{C}_I)Q = -2\hat{u}Q, \quad A_1 = 0, \quad A_2 = -\hat{u}Q^2 .$$

Then (3.7) follows from these and (3.2).

Note that although the derivation is different equation (3.7) is the same as (7.37) of [56] if we make the change:  $\Delta = \ell + 2$ .

### 3.2 Two-point Green function in bulk

We need also the two-point Green function in bulk. Standardly for this we derive the Green function of the operator  $\mathcal{C} - \lambda$

$$(\mathcal{C} - \lambda)G(x, y, \zeta; x', y', \zeta') = y^4 \delta^3(x - x') \delta(y - y') (\zeta, \zeta')^s. \quad (3.9)$$

The computation of  $G$  is more or less same as the one for eigenvalue problem of  $\mathcal{C}$  in the previous subsection. We assume  $G$  has a factored form

$$G(x, y, \zeta; x', y', \zeta') = f(u) Q(x, y, \zeta; x', y', \zeta')^s,$$

where  $u$  is the  $SO(3, 2)$  invariant of two points  $(x, y)$  and  $(x', y')$  given in (2.19). This assumption is justified a posteriori since the two-point function is unique up to multiplicative constant.

Note that  $\mathbf{G}$  is a eigenfunction of  $\mathcal{C}_I$ , i.e.,  $\mathcal{C}_I \mathbf{G} = \lambda_I \mathbf{G}$ . Then  $\mathbf{G}$  is given by

$$\begin{aligned} \mathbf{G} &= u^\Delta F(u) Q^s, \\ Q &= \frac{2 I'_1 - ((\mathbf{x} - \mathbf{x}')^2 - (y + y')^2) I_2 - 2(y + y') I'_3}{(\mathbf{x} - \mathbf{x}')^2 + (y + y')^2}, \\ I'_1 &= \langle x - x', \zeta \rangle \langle x - x', \zeta' \rangle, \quad I_2 = \langle \zeta, \zeta' \rangle, \quad I'_3 = \sum_{\mu=0}^2 (x_\mu - x'_\mu) (\zeta \times \zeta')_\mu, \end{aligned} \quad (3.10)$$

and  $F(u)$  is a singular solution of the hypergeometric equation

$$\left( u(1-u) \frac{d^2}{du^2} + 2[\Delta - 1 - \Delta u] \frac{d}{du} + (s - \Delta + 1)(s + \Delta) \right) F(u) = 0. \quad (3.11)$$

We sketch the derivation of the Green function. First we observe that

$$(\mathcal{C} - \mathcal{C}_I) \mathbf{G} = \mathbf{G} [f^{-1} \mathcal{C}_B f + s Q^{-1} (\mathcal{C} - \mathcal{C}_I) Q + 2s f^{-1} Q^{-1} A_1 + s(s-1) Q^{-2} A_2], \quad (3.12)$$

where

$$\begin{aligned} A_1 &= -y^2 (\langle \partial f, \partial Q \rangle + (\partial_y f)(\partial_y Q)) - y ((\partial_0 f) \mathfrak{s}_{12} Q - (\partial_1 f) \mathfrak{s}_{02} Q + (\partial_2 f) \mathfrak{s}_{01} Q), \\ A_2 &= -y^2 (\langle \partial Q, \partial Q \rangle + (\partial_y Q)^2) - 2y ((\partial_0 Q) \mathfrak{s}_{12} Q - (\partial_1 Q) \mathfrak{s}_{02} Q + (\partial_2 Q) \mathfrak{s}_{01} Q). \end{aligned}$$

Straightforward computation shows that

$$(\mathcal{C} - \mathcal{C}_I) Q = -2uQ, \quad A_1 = 0, \quad A_2 = -uQ^2.$$

It follows that

$$(\mathcal{C} - \mathcal{C}_I) \mathbf{G} = Q^s \left( u^2(u-1) \frac{d^2}{du^2} + 2u \frac{d}{du} - s(s+1)u \right) f(u).$$

Setting  $f(u) = u^\Delta F(u)$  we have

$$(\mathcal{C} - \lambda) \mathbf{G} = -Q^s u^{\Delta+1} \left( u(1-u) \frac{d^2}{du^2} + 2[\Delta - 1 - \Delta u] \frac{d}{du} + (s - \Delta + 1)(s + \Delta) \right) F(u).$$

Thus if  $F(u)$  is a singular solution of the hypergeometric equation then we obtain the RHS of (3.9). The delta functions in the RHS corresponds to the singularity at  $u = 1 \Leftrightarrow x_\mu = x'_\mu, y = y'$ .

**Remark 1:** One may wonder whether the above may be generalised to the anti de Sitter algebra  $so(d, 2)$  for  $d > 3$ . Actually, the only difficulty for  $d > 3$  and nontrivial spin would be to find the explicit form of the function  $Q$ . In fact, below some calculations are valid implicitly or explicitly for arbitrary  $d$ .

## 4 Bulk-boundary correspondence

Consider the fields  $\varphi \in C^\Lambda$  and  $\phi \in \hat{C}^\Lambda$  from the boundary and bulk representations for the same  $\Lambda$ . By construction they are eigenfunctions of the Casimir operator with the same eigenvalue:

$$\mathcal{C}\varphi = \lambda\varphi, \quad \mathcal{C}\phi = \lambda\phi. \quad (4.1)$$

The bulk field behaves as in (2.18) when approaching the boundary. Thus, we define the bulk-to-boundary operator  $L_\Delta$  by:

$$\begin{aligned} L_\Delta : \hat{C}^\Lambda &\longrightarrow C^\Lambda \\ \varphi(x, \zeta) &= (L_\Delta \phi)(x, \zeta) := \lim_{y \rightarrow 0} y^{-\Delta} \phi(x, y, \zeta). \end{aligned} \quad (4.2)$$

On the other hand the boundary-to-bulk operator  $\tilde{L}_\Lambda$  is defined by:

$$\begin{aligned} \tilde{L}_\Lambda : C^\Lambda &\longrightarrow \hat{C}^\Lambda \\ \phi(x, y, \zeta) &= (\tilde{L}_\Lambda \varphi)(x, \zeta) := \int S_\Lambda(x - x', y; \zeta, \partial_{\zeta'}) \varphi(x', \zeta') d^3 x' \end{aligned} \quad (4.3)$$

where the kernel  $S_\Lambda$  is obtained from the two-point Green function  $\mathbf{G}$  defined in (3.9) as follows

$$S_\Lambda(x - x', y; \zeta, \partial_{\zeta'}) = \lim_{y' \rightarrow 0} y'^{\Delta-3} \mathbf{G}(x, y, \zeta; x', y', \partial_{\zeta'}). \quad (4.4)$$

The formula for  $S_\Lambda$  (for  $s \in \mathbb{Z}_+$ ) is given by

$$S_\Lambda = N_\Lambda \tilde{u}^{3-\Delta} R^s, \quad \tilde{u} = \frac{4y}{(\mathbf{x} - \mathbf{x}')^2 + y^2}, \quad R = \frac{\tilde{u}\mathcal{L}}{4y}, \quad (4.5)$$

with

$$\mathcal{L} = 2\tilde{I}_1 - ((\mathbf{x} - \mathbf{x}')^2 - y^2)\tilde{I}_2 - 2y\tilde{I}_3, \quad (4.6)$$

$$\tilde{I}_1 = \langle x - x', \zeta \rangle \langle x - x', \partial_{\zeta'} \rangle, \quad \tilde{I}_2 = \langle \zeta, \partial_{\zeta'} \rangle, \quad \tilde{I}_3 = \sum_{\mu=0}^2 (x_\mu - x'_\mu) (\zeta \times \partial_{\zeta'})_\mu,$$

and  $N_\Lambda$  is a normalization constant depending on the representation  $\Lambda = [s, \Delta]$ .

Now we check consistency of the operators  $L_\Delta$  and  $\tilde{L}_\Lambda$ :

$$L_\Delta \circ \tilde{L}_\Lambda = \mathbf{1}_{C^\Lambda}, \quad \tilde{L}_\Lambda \circ L_\Delta = \mathbf{1}_{\hat{C}^\Lambda}. \quad (4.7)$$

For the first relation in (4.7) we have to show that:

$$\begin{aligned} \varphi(x, \zeta) &= (L_\Delta \circ \tilde{L}_\Lambda \varphi)(x, \zeta) \\ &= \lim_{y \rightarrow 0} y^{-\Delta} \int S_\Lambda(x - x', y; \zeta, \partial_{\zeta'}) \varphi(x', \zeta') d^3 x'. \end{aligned} \quad (4.8)$$

We take the limit first by exchanging it and the integral. To calculate the limit it is necessary to express the kernel  $S_\Lambda$  in another form. To this end we establish the following formula of Fourier transform:

$$\int \frac{e^{i\langle p, X \rangle}}{(\langle X, X \rangle + y^2)^\alpha} \frac{d^3 X}{(2\pi)^{3/2}} = \frac{i\pi}{(-1)^{2\alpha-1} 2^\alpha \Gamma(\alpha)} \left( \frac{\sqrt{-\mathbf{p}^2}}{y} \right)^{\alpha-3/2} H_{\alpha-3/2}^{(1)}(y\sqrt{-\mathbf{p}^2}), \quad (4.9)$$

where  $X_\mu = x_\mu - x'_\mu$  and  $H_\beta^{(1)}$  is a Hankel function. The  $(X_1, X_2)$  integration can be carried out by making use of the following two formulae: First one is a formula for  $(d-1)$  dimensional angular integration in  $d$  dimensional Euclidean space:

$$\int f(r) e^{-i\vec{p} \cdot \vec{x}} \frac{d^d x}{(2\pi)^d} = \left( \frac{1}{2\pi} \right)^{d/2} \int_0^\infty r f(r) \left( \frac{r}{p} \right)^{\frac{d}{2}-1} J_{\frac{d}{2}-1}(pr) dr, \quad (4.10)$$

$$\vec{p} = (p_1, p_2, \dots, p_d), \quad \vec{x} = (x_1, x_2, \dots, x_d), \quad p^2 = \sum_{k=1}^d p_k^2, \quad r^2 = \sum_{k=1}^d x_k^2,$$

which is valid for any radial function  $f(r)$ . Second formula is an integration of Bessel function:

$$\int_0^\infty \frac{r^{\beta+1} J_\beta(ar)}{(r^2 + \rho^2)^{\gamma+1}} dr = \frac{a^\gamma \rho^{\beta-\gamma} K_{\beta-\gamma}(a\rho)}{2^\gamma \Gamma(\gamma+1)}, \quad 2\Re \gamma + \frac{3}{2} > \Re \beta > -1. \quad (4.11)$$

We modify the second formula (4.11). Set  $\beta = 0$  and replace  $\rho$  with  $-i\rho$ , then use the relation between Bessel functions

$$K_\gamma(z) = \frac{\pi}{2} i e^{\gamma\pi i/2} H_\gamma^{(1)}(iz), \quad -\pi < \arg z < \frac{\pi}{2} \quad (4.12)$$

we obtain

$$\int_0^\infty \frac{r J_0(ar)}{(r^2 - \rho^2)^{\gamma+1}} dr = \frac{i\pi a^\gamma}{(-1)^\gamma 2^{\gamma+1} \Gamma(\gamma+1) \rho^\gamma} H_\gamma^{(1)}(a\rho). \quad (4.13)$$

Now we return to the Fourier transform (4.9). Angular integration in  $X_1 X_2$  plane is performed by (4.10) and we use (4.13) for the radial integration in the plane:

$$\begin{aligned} \int \frac{e^{i\langle p, X \rangle}}{(\langle X, X \rangle + y^2)^\alpha} \frac{d^3 X}{(2\pi)^{3/2}} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty dX_0 \int_0^\infty dr \frac{r J_0(\tilde{p}r)}{(-1)^\alpha (r^2 - X_0^2 - y^2)^\alpha} e^{ip_0 X_0} \\ &= \frac{i\pi}{(-1)^{2\alpha-1} \sqrt{2\pi} \Gamma(\alpha)} \left( \frac{\tilde{p}}{2} \right)^{\alpha-1} \int_0^\infty dX_0 \frac{H_{\alpha-1}^{(1)}(\tilde{p}\sqrt{X_0^2 + y^2})}{(X_0^2 + y^2)^{(\alpha-1)/2}} \cos p_0 X_0 \end{aligned}$$

where

$$r^2 = X_1^2 + X_2^2, \quad \tilde{p}^2 = p_1^2 + p_2^2.$$

Recalling that

$$H_\beta^{(1)}(z) = J_\beta(z) + iY_\beta(z)$$

$X_0$  integration is performed by the formulae of Fourier cosine transform [64]

$$\begin{array}{c|c} f(r) & \int_0^\infty f(r) \cos(r\rho) dr \\ \hline \frac{J_\beta(a\sqrt{r^2+b^2})}{(r^2+b^2)^{\beta/2}} & \begin{cases} \sqrt{\frac{\pi b}{2}} \frac{(a^2-\rho^2)^{\beta/2-1/4}}{(ab)^\beta} J_{\beta-1/2}(b\sqrt{a^2-\rho^2}) & 0 < \rho < a \\ 0 & a < \rho \end{cases} \\ \hline \frac{Y_\beta(a\sqrt{r^2+b^2})}{(r^2+b^2)^{\beta/2}} & \begin{cases} \sqrt{\frac{\pi b}{2}} \frac{(a^2-\rho^2)^{\beta/2-1/4}}{(ab)^\beta} Y_{\beta-1/2}(b\sqrt{a^2-\rho^2}) & 0 < \rho < a \\ -\sqrt{\frac{2b}{\pi}} \frac{(\rho^2-a^2)^{\beta/2-1/4}}{(ab)^\beta} K_{\beta-1/2}(b\sqrt{\rho^2-a^2}) & a < \rho \end{cases} \end{array}$$

$$\Re \beta > -\frac{1}{2}, \quad a, b > 0$$

By these formula we obtain

$$\begin{aligned} & \int_0^\infty dX_0 \frac{H_{\alpha-1}^{(1)}(\tilde{p}\sqrt{X_0^2+y^2})}{(X_0^2+y^2)^{(\alpha-1)/2}} \cos(p_0 X_0) \\ &= \begin{cases} \left(\frac{\pi y}{2}\right)^{1/2} \frac{(-\langle p, p \rangle)^{\alpha/2-3/4}}{(\tilde{p}y)^{\alpha-1}} H_{\alpha-3/2}^{(1)}(y\sqrt{\tilde{p}^2-p_0^2}) & 0 < p_0 < \tilde{p} \\ -i \left(\frac{2y}{\pi}\right)^{1/2} \frac{\langle p, p \rangle^{\alpha/2-3/4}}{(\tilde{p}y)^{\alpha-1}} K_{\alpha-3/2}(y\sqrt{p_0^2-\tilde{p}^2}) & \tilde{p} < p_0 \end{cases} \\ &= \left(\frac{\pi y}{2}\right)^{1/2} \frac{(-\langle p, p \rangle)^{\alpha/2-3/4}}{(\tilde{p}y)^{\alpha-1}} H_{\alpha-3/2}^{(1)}(y\sqrt{\tilde{p}^2-p_0^2}). \end{aligned} \quad (4.14)$$

In the last equality the relation (4.12) was used to unify two separate cases. Note that  $\langle p, p \rangle = p_0^2 - \tilde{p}^2$ . In this way the Fourier transform (4.9) has been established.

Now we evaluate the Fourier transform of the kernel  $S_\Lambda$

$$\begin{aligned} & \int S_\Lambda(X, y; \zeta, \partial_{\zeta'}) e^{i\langle p, X \rangle} \frac{d^3 X}{(2\pi)^{3/2}} = N_\Lambda \int \frac{(4y)^{3-\Delta}}{(\langle X, X \rangle + y^2)^{s-\Delta+3}} \mathbf{S}^s e^{i\langle p, X \rangle} \frac{d^3 X}{(2\pi)^{3/2}} \\ &= \frac{-i\pi N_\Lambda}{2^{s+\Delta-1}\Gamma(s-\Delta+3)y^{s-3/2}} \mathbf{S}^s (\sqrt{-\mathbf{p}^2})^{s-\Delta+3/2} H_{s-\Delta+3/2}^{(1)}(y\sqrt{-\mathbf{p}^2}), \end{aligned}$$

where

$$\mathbf{S} = -2(\partial_p \cdot \zeta) \partial_p \cdot \partial_{\zeta'} + (\langle \partial_p, \partial_p \rangle + y^2) \langle \zeta, \partial_{\zeta'} \rangle + 2iy \langle \partial_p, \zeta \times \partial_{\zeta'} \rangle,$$

with  $a \cdot b = \sum_{\mu=0}^2 a^\mu b_\mu$ . Inverse Fourier transform gives the following formula of the kernel

$$S_\Lambda = \frac{-i\pi N_\Lambda}{2^{s+\Delta-1}\Gamma(s-\Delta+3)y^{s-3/2}} \int \mathbf{S}^s(\sqrt{-\mathbf{p}^2})^{s-\Delta+3/2} H_{s-\Delta+3/2}^{(1)}(y\sqrt{-\mathbf{p}^2}) e^{-i\langle \mathbf{p}, X \rangle} \frac{d^3 p}{(2\pi)^{3/2}}. \quad (4.15)$$

Since we take a limit of  $y \rightarrow 0$ , we replace the Hankel function with its asymptotic form

$$-iH_\alpha^{(1)}(z) \rightarrow -\frac{\Gamma(\alpha)}{\pi} \left(\frac{2}{z}\right)^\alpha, \quad z \rightarrow 0$$

Then

$$-i(\sqrt{-\mathbf{p}^2})^{s-\Delta+3/2} H_{s-\Delta+3/2}^{(1)}(y\sqrt{-\mathbf{p}^2}) = -\frac{\Gamma(s-\Delta+3/2)}{\pi} \left(\frac{2}{y}\right)^{s-\Delta+3/2}, \quad s-\Delta+3/2 \notin \mathbb{Z}_-,$$

is independent of  $p_\mu$  so that the action of  $\mathbf{S}$  is reduced to  $y^2 \langle \zeta, \partial_{\zeta'} \rangle$  and the integration over  $p$  becomes Dirac's delta function:

$$S_\Lambda \rightarrow -\frac{(2\pi)^{3/2} N_\Lambda \Gamma(s-\Delta+3/2)}{2^{2\Delta-5/2} \Gamma(s-\Delta+3)} y^\Delta \delta^3(X) \langle \zeta, \partial_{\zeta'} \rangle^s, \quad s-\Delta+3/2 \notin \mathbb{Z}_-, \quad y \rightarrow 0. \quad (4.16)$$

Substituting this formula of  $S$  in (4.8) we obtain:

$$\begin{aligned} \varphi(x, \zeta) &= -\frac{\pi^{3/2} N_\Lambda \Gamma(s-\Delta+3/2)}{2^{2\Delta-4} \Gamma(s-\Delta+3)} \varphi(x, \zeta), \\ s-\Delta+3/2 &\notin \mathbb{Z}_-, \quad s-\Delta+3 \notin \mathbb{Z}_-. \end{aligned} \quad (4.17)$$

From the latter we see the first consistency relation (4.7) being true by an appropriate choice of  $N_\Lambda$ , e.g.

$$\begin{aligned} N_\Lambda &= -\frac{2^{2\Delta-4} \Gamma(s-\Delta+3)}{\pi^{3/2} \Gamma(s-\Delta+3/2)}, \\ s-\Delta+3/2 &\notin \mathbb{Z}_-, \quad s-\Delta+3 \notin \mathbb{Z}_-. \end{aligned} \quad (4.18)$$

As a *Corollary* we conclude that for generic values of  $\Delta$  we can reconstruct a function on anti de Sitter space from its boundary value. Indeed, suppose we have:

$$\phi(x, y, \zeta) = \int S_\Lambda(x-x', y; \zeta, \partial_{\zeta'}) f(x', \zeta') dx'. \quad (4.19)$$

Then we have for the boundary value:

$$\begin{aligned} \psi_0(x, \zeta) &\doteq (L_\Delta \phi)(x, \zeta) = \lim_{y \rightarrow 0} y^{-\Delta} \phi(x, y, \zeta) = \\ &= \lim_{y \rightarrow 0} y^{-\Delta} \int S_\Lambda(x-x', y; \zeta, \partial_{\zeta'}) f(x', \zeta') dx' = f(x) \end{aligned} \quad (4.20)$$

Now we can prove the second consistency relation in (4.7):

$$\begin{aligned}
(\tilde{L}_\Lambda \circ L_\Delta \phi)(x, y, \zeta) &= \int S_\Lambda(x - x', y; \zeta, \partial_{\zeta'}) (L_\Delta \phi)(x', \zeta') dx' = \\
&= \int S_\Lambda(x - x', y; \zeta, \partial_{\zeta'}) \lim_{y' \rightarrow 0} y'^{-\Delta} \phi(x', y', \zeta') = \\
&= \int S_\Lambda(x - x', y; \zeta, \partial_{\zeta'}) \psi_0(x', \zeta') dx' = \phi(x, y, \zeta) \quad (4.21)
\end{aligned}$$

where in the last line we used (4.20).

## 5 Intertwining properties

Here we investigate the intertwining properties of the boundary  $\leftrightarrow$  bulk operators.

### 5.1 Bulk-to-boundary operator $L_\Delta$

It is not difficult to verify the intertwining property of the Bulk-to-boundary operator  $L_\Delta$ . Namely, one should verify the following:

$$L_\Delta \circ \hat{X} = \tilde{X} \circ L_\Delta, \quad (5.1)$$

where  $X \in so(3, 2)$ ,  $\tilde{X}$  denotes the action of the generator  $X$  on the boundary (2.7) and  $\hat{X}$  denotes the action of the generator in the bulk (2.13). More explicitly,

$$\tilde{X} \varphi(x, \zeta) = \lim_{y \rightarrow 0} y^{-\Delta} \hat{X} \phi(x, y, \zeta), \quad \varphi \in C^\Lambda, \quad \phi \in \hat{C}^\Lambda \quad (5.2)$$

If the field  $\varphi$  belongs to the conjugate representation  $\varphi \in C^{\tilde{\Lambda}}$ ,  $\tilde{\Lambda} = [s, 3 - \Delta]$ , then relations (5.1), (5.2) hold with the change  $\Delta \rightarrow 3 - \Delta$ , the same change being made also in (2.7).

### 5.2 Boundary-to-bulk operator $\tilde{L}_\Lambda$

The intertwining property of the boundary-to-bulk operator  $\tilde{L}_\Lambda$  means that

$$\hat{X} \circ \tilde{L}_\Lambda = \tilde{L}_\Lambda \circ \tilde{X} \quad (5.3)$$

More explicitly, it reads

$$\hat{X} \phi(x, y, \zeta) = \int S_\Lambda(x, y, \zeta; x', \partial_{\zeta'}) \tilde{X}_\Lambda \varphi(x', \zeta') d^3 x', \quad \phi \in \hat{C}^\Lambda, \quad \varphi \in C^\Lambda \quad (5.4)$$

This is an immediate consequence of  $L_\Delta \circ \tilde{L}_\Lambda = \mathbf{1}_{C^\Lambda}$ ,  $\tilde{L}_\Lambda \circ L_\Delta = \mathbf{1}_{\hat{C}^\Lambda}$  and (5.1). By sandwiching (5.1) by  $\tilde{L}_\Lambda$  one has

$$\tilde{L}_\Lambda \circ L_\Delta \circ \hat{X} \circ \tilde{L}_\Lambda = \tilde{L}_\Lambda \circ \tilde{X} \circ L_\Delta \circ \tilde{L}_\Lambda, \quad \text{acting on } C^\Lambda$$

This is nothing but (5.3).

**Proof of (5.4) by direct computation**

A key observation to check the intertwining property is the following identities:

$$\frac{\partial \tilde{u}}{\partial x_\mu} = -\frac{\partial \tilde{u}}{\partial x'_\mu}, \quad \frac{\partial R}{\partial x_\mu} = -\frac{\partial R}{\partial x'_\mu}, \quad \mu = 0, 1, 2 \quad (5.5)$$

It follows that

$$\frac{\partial S_\Lambda}{\partial x_\mu} = -\frac{\partial S_\Lambda}{\partial x'_\mu}, \quad \mu = 0, 1, 2 \quad (5.6)$$

Formulas of differentiation by  $y$  :

$$\frac{\partial \tilde{u}}{\partial y} = \frac{\tilde{u}}{y} - \frac{\tilde{u}^2}{2}, \quad \frac{\partial R}{\partial y} = \frac{\tilde{u}}{4y} \partial_y \mathcal{L} - \frac{\tilde{u}R}{2}.$$

It follows that

$$\begin{aligned} \frac{\partial S_\Lambda}{\partial y} &= \left( \frac{3 - \Delta}{y} - (s - \Delta + 3) \frac{\tilde{u}}{2} + s \frac{1}{\mathcal{L}} \partial_y \mathcal{L} \right) S_\Lambda, \\ \left( \sum_{\mu=0}^2 x_\mu \frac{\partial}{\partial x_\mu} + y \frac{\partial}{\partial y} \right) S_\Lambda &= - \left( 3 - \Delta + \sum_{\mu=0}^2 x'_\mu \frac{\partial}{\partial x'_\mu} \right) S_\Lambda. \end{aligned} \quad (5.7)$$

With this identity, it is immediate to verify the intertwining property for  $T_\mu$  and  $D$ .



### 5.3 Further intertwining relations

We start by recording the second limit of the bulk functions

$$\begin{aligned}
\varphi_0(x, \zeta) &\doteq \lim_{y \rightarrow 0} y^{\Delta-3} \phi(x, y, \zeta) = \\
&= \lim_{y \rightarrow 0} y^{\Delta-3} \int S_\Lambda(x - x', y; \zeta, \partial_{\zeta'}) \psi_0(x', \zeta') dx' = \\
&= N_\Lambda \lim_{y \rightarrow 0} y^{\Delta-3} \int \left( \frac{4y}{(\mathbf{x} - \mathbf{x}')^2 + y^2} \right)^{3-\Delta} \left( \frac{\mathcal{L}}{(\mathbf{x} - \mathbf{x}')^2 + y^2} \right)^s \psi_0(x', \zeta') dx' \\
&= N_\Lambda \int \left( \frac{4}{(\mathbf{x} - \mathbf{x}')^2} \right)^{3-\Delta} \left( \frac{2\tilde{I}_1 - ((\mathbf{x} - \mathbf{x}')^2)\tilde{I}_2}{(\mathbf{x} - \mathbf{x}')^2} \right)^s \psi_0(x', \zeta') dx' = \\
&= \mathcal{N}_\Lambda \int \frac{dx'}{((\mathbf{x} - \mathbf{x}')^2)^{3-\Delta}} \left( \frac{2\langle x - x', \zeta \rangle \langle x - x', \partial_{\zeta'} \rangle}{(\mathbf{x} - \mathbf{x}')^2} - \langle \zeta, \partial_{\zeta'} \rangle \right)^s \psi_0(x', \zeta') \\
&= \mathcal{N}_\Lambda \int \frac{dx'}{((\mathbf{x} - \mathbf{x}')^2)^{3-\Delta}} (r(x - x'; \zeta, \partial_{\zeta'}))^s \psi_0(x', \zeta') = \\
&= \frac{\mathcal{N}_\Lambda}{\gamma_{\tilde{\Lambda}}} \int dx' G_{\tilde{\Lambda}}(x - x'; \zeta, \partial_{\zeta'}) \psi_0(x', \zeta'), \quad \mathcal{N}_\Lambda = 4^{3-\Delta} N_\Lambda,
\end{aligned} \tag{5.8}$$

where in the second line we have used (4.20), in the third line we have used (4.5) and (4.6), and in the last line we have recovered the well-known conformal two-point function, cf., e.g., [65]:

$$\begin{aligned}
G_\Lambda(x; \zeta, \zeta') &= \gamma_\Lambda \frac{(r(x; \zeta, \zeta'))^s}{(\mathbf{x}^2)^\Delta}, \\
r(x; \zeta, \zeta') &= r(x)_{\mu\sigma} \zeta^\mu \zeta'^\sigma, \quad r(x)_{\mu\sigma} = \frac{2}{\mathbf{x}^2} x_\mu x_\sigma - g_{\mu\sigma} \\
g &= (g_{\mu\nu}) = \text{diag}(1, -1, -1)
\end{aligned} \tag{5.9}$$

for the conjugate weight  $\tilde{\Lambda} = [s, 3 - \Delta]$ . The latter is natural since  $\psi_0 \in C^\Lambda$ ,  $\varphi_0 \in C^{\tilde{\Lambda}}$ , and the conformal two-point function realizes the equivalence of the conjugate representations  $\Lambda, \tilde{\Lambda}$  which have the same Casimir values, cf. [56]. The normalization constant  $\gamma_\Lambda$  depends on the representation  $\Lambda = [s, \Delta]$  and below we derive a formula for the product  $\gamma_\Lambda \gamma_{\tilde{\Lambda}}$ .

Further, using (5.8) we define the operator  $G_\Lambda$  through the kernel  $G_\Lambda(x; \zeta, \zeta')$ :

$$\begin{aligned}
G_\Lambda &: C^{\tilde{\Lambda}} \rightarrow C^\Lambda, \\
(G_\Lambda \varphi_0)(x, \zeta) &= \int dx' G_\Lambda(x - x'; \zeta, \partial_{\zeta'}) \varphi_0(x', \zeta').
\end{aligned} \tag{5.10}$$

Then relation (5.8) may be written as:

$$L_{\tilde{\Delta}} = \frac{\mathcal{N}_\Lambda}{\gamma_{\tilde{\Lambda}}} G_{\tilde{\Lambda}} \circ L_\Delta, \quad \tilde{\Delta} \doteq 3 - \Delta \tag{5.11}$$

as operators acting on the bulk representation  $\hat{C}^\Lambda$ .

Note that at generic points (those not excluded in (4.17)) the operators  $G_\Lambda$  and  $G_{\bar{\Lambda}}$  are inverse to each other [56]:

$$G_\Lambda \circ G_{\bar{\Lambda}} = \mathbf{1}_{C^\Lambda}, \quad G_{\bar{\Lambda}} \circ G_\Lambda = \mathbf{1}_{C^{\bar{\Lambda}}}. \quad (5.12)$$

At generic points from this we can obtain a lot of interesting relations, e.g., applying  $\tilde{L}_\Lambda$  from the right we get:

$$L_{\bar{\Delta}} \circ \tilde{L}_\Lambda = \frac{\mathcal{N}_\Lambda}{\gamma_{\bar{\Lambda}}} G_{\bar{\Lambda}} \quad (5.13)$$

Then we write down the conjugate relation:

$$L_\Delta \circ \tilde{L}_{\bar{\Lambda}} = \frac{\mathcal{N}_{\bar{\Lambda}}}{\gamma_\Lambda} G_\Lambda \quad (5.14)$$

Then we combine relations (5.13) and (5.14):

$$L_\Delta \circ \tilde{L}_{\bar{\Lambda}} \circ L_{\bar{\Delta}} \circ \tilde{L}_\Lambda = \frac{\mathcal{N}_{\bar{\Lambda}}}{\gamma_\Lambda} \frac{\mathcal{N}_\Lambda}{\gamma_{\bar{\Lambda}}} G_\Lambda \circ G_{\bar{\Lambda}} = \frac{\mathcal{N}_{\bar{\Lambda}}}{\gamma_\Lambda} \frac{\mathcal{N}_\Lambda}{\gamma_{\bar{\Lambda}}} \mathbf{1}_\Lambda \quad (5.15)$$

For the LHS of (5.15) we use first the second relation of (4.7), then the first to obtain:

$$L_\Delta \circ \tilde{L}_{\bar{\Lambda}} \circ L_{\bar{\Delta}} \circ \tilde{L}_\Lambda = L_\Delta \circ \mathbf{1}_{\hat{C}^\Lambda} \circ \tilde{L}_\Lambda = L_\Delta \circ \tilde{L}_\Lambda = \mathbf{1}_\Lambda. \quad (5.16)$$

Thus, from (5.15) and (5.16) follows:

$$\begin{aligned} \gamma_\Lambda \gamma_{\bar{\Lambda}} &= \mathcal{N}_\Lambda \mathcal{N}_{\bar{\Lambda}} = \frac{2^4 \Gamma(s - \Delta + 3) \Gamma(s + \Delta)}{\pi^3 \Gamma(s - \Delta + 3/2) \Gamma(s + \Delta - 3/2)}, \\ s - \Delta + 3/2 &\notin \mathbb{Z}_-, \quad s - \Delta + 3 \notin \mathbb{Z}_-, \\ s + \Delta - 3/2 &\notin \mathbb{Z}_-, \quad s + \Delta \notin \mathbb{Z}_-. \end{aligned} \quad (5.17)$$

The product of constants in (5.17) should be proportional to the analytic continuation of the Plancherel measure for the Plancherel formula contribution of the principal series of unitary irreps of  $G$ , cf., e.g., [28], but we shall not go into that.

**Remark 2:** One may wonder what happens at the excluded values in (5.17). This requires further nontrivial examination. Such study was done in the Euclidean case in [56]. Since some results may follow by Wick rotation we may conjecture that, for example, the operators  $G_\Lambda$  and  $G_{\bar{\Lambda}}$  would not be inverse to each other. This would be since at these points the representations  $C^\Lambda$  and  $C^{\bar{\Lambda}}$  would be reducible and the  $G$ -operators would have kernels. All such properties are currently under study [66].

## References

- [1] J. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231 (hep-th/971120).
- [2] M. Flato and C. Fronsdal, J. Math. Phys. **22** (1981) 1100.
- [3] E. Angelopoulos, M. Flato, C. Fronsdal and D. Sternheimer, Phys. Rev. **D23** (1981) 1278.
- [4] C. Fronsdal, Phys. Rev. **D26** (1982) 1988.
- [5] P. Breitenlohner and D.Z. Freedman, Phys. Lett. **B115** (1982) 197.
- [6] H. Nicolai and E. Sezgin, Phys. Lett. **143B** (1984) 103.
- [7] S. Ferrara and C. Fronsdal, Class. Quant. Grav. **15** (1998) 2153;
- [8] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Phys. Lett. **B428** (1998) 105.
- [9] E. Witten, Adv. Theor. Math. Phys. **2** (1998) 253.
- [10] S. Ferrara, C. Fronsdal and A. Zaffaroni, Nucl. Phys. **B532** (1998) 153; S. Ferrara and C. Fronsdal, Phys. Lett. **B433** (1998) 19; Lett. Math. Phys. **44** (1998) 249; Lett. Math. Phys. **46** (1998) 157.
- [11] Y. Oz and J. Terning, Nucl. Phys. **B532** (1998) 163.
- [12] L. Andrianopoli and S. Ferrara, Phys. Lett. **B430** (1998) 248; Lett. Math. Phys. **46** (1998) 265; Lett. Math. Phys. **48** (1999) 145; S. Ferrara, M.A. Lledo and A. Zaffaroni, Phys. Rev. **D58** (1998) 105029; S. Ferrara, M. Porrati and A. Zaffaroni, Lett. Math. Phys. **47** (1999) 255.
- [13] M. Henningson and K. Sfetsos, Phys. Lett. **B431** (1998) 63.
- [14] W. Mück and K.S. Viswanathan, Phys. Rev. **D58** (1998) 041901; Phys. Rev. **D58** (1998) 106006; Phys. Rev. **D60** (1999) 081901.
- [15] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, Nucl. Phys. **B546** (1999) 96; Phys. Lett. **B452** (1999) 61; E. D'Hoker and D.Z. Freedman, Nucl. Phys. **B544** (1999) 612; Nucl. Phys. **B550** (1999) 261.
- [16] H. Liu and A.A. Tseytlin, Nucl. Phys. **B533** (1998) 88; Phys. Rev. **D59** (1999) 086002.
- [17] G. Chalmers, H. Nastase, K. Schalm and R. Siebelink, Nucl. Phys. **B540** (1999) 247.
- [18] V. Balasubramanian, P. Kraus and A.E. Lawrence, Phys. Rev. **D59** (1999) 046003.
- [19] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, Adv. Theor. Math. Phys. **2** (1998) 697.
- [20] M.J. O'Loughlin and S. Randjbar-Daemi, Nucl. Phys. **B543** (1999) 170.
- [21] S. Corley, Phys. Rev. **D59** (1999) 086003.
- [22] A. Volovich, J. High Energy Phys. **9809** (1998) 022.
- [23] D. Berenstein, R. Corrado, W. Fischler and J. Maldacena, Phys. Rev. **D59** (1999) 105023.
- [24] G. Bonelli, Phys. Lett. **B450** (1999) 363.
- [25] M. Bianchi and S. Kovacs, ROM2F-98-37, hep-th/9811060; Phys. Lett. **B468** (1999) 102.
- [26] W.-S. Li, Phys. Lett. **B448** (1999) 218.
- [27] H. Liu, Phys. Rev. **D60** (1999) 106005.
- [28] V.K. Dobrev, Nucl. Phys. **B553** [PM] (1999) 559.
- [29] A.M. Ghezelbash, K. Kaviani, S. Parvizi and A.H. Fatollahi, Phys. Lett. **B435** (1998) 291; A.M. Ghezelbash, M. Khorrami and A. Aghamohammadi, Int. J. Mod. Phys. **A14** (1999) 2581; K. Kaviani and A.M. Ghezelbash, Phys. Lett. **B469** (1999) 81; K. Kaviani, Int. J. Mod. Phys. **B14** (2000) 2355.

- [30] R.C. Rashkov, Mod. Phys. Lett. **A14** (1999) 1783; Phys. Lett. **B466** (1999) 190.
- [31] K.-H. Rehren, Ann. Henri Poincare **1** (4) (2000) 607; in: SFIN XVIII Ser. A, eds. B. Dragovich et al., A1 (2005) 95; M. Duetsch and K.H. Rehren, Ann. Henri Poincare **4** (2003) 613; Lett. Math. Phys. **62** (2002) 171.
- [32] I.R. Klebanov and E. Witten, Nucl. Phys. **B556** (1999) 89.
- [33] R.R. Metsaev, Nucl. Phys. **B563** (1999) 295 ; Class. Quant. Grav. **22** (2005) 2777; Phys. Lett. **B636** (2006) 227; J. High Energy Phys. **1201** (2012) 064; J. High Energy Phys. **1206** (2012) 062; Phys. Rev. **D78** (2008) 106010; Phys.Rev. **D81** (2010) 106002.
- [34] P. Mincses and V.O. Rivelles, Nucl. Phys. **B572** (2000) 651; J. High Energy Phys. **0112** (2001) 010; P. Mincses, Phys. Rev. **D70** (2004) 025011.
- [35] K. Zoubos, J. High Energy Phys. **0212** (2002) 037; J. High Energy Phys. **0501** (2005) 031.
- [36] S. Deser and A. Waldron, Nucl. Phys. **B662** (2003) 379; S. Carlip, S. Deser, A. Waldron and D.K. Wise, Class. Quant. Grav. **26** (2009) 075008.
- [37] R.G. Leigh and A.C. Petkou, J. High Energy Phys. **0306** (2003) 011; J. High Energy Phys. **0312** (2003) 020.
- [38] T. Leonhardt, R. Manvelyan and W. Ruehl, Nucl. Phys. **B667** (2003) 413; J. Phys. **A37** (2004) 7051; W. Ruehl, Nucl. Phys. **B705** (2005) 437; Phys. Lett. **B605** (2005) 413; Phys. Atom. Nucl. **75** (2012) 1279; R. Manvelyan and W. Ruehl, Phys. Lett. **B613** (2005) 197; Nucl. Phys. **B717** (2005) 3.
- [39] E. Witten, in: *From Fields to Stings: Circumnavigating Theoretical Physics*, M. Shifman, A. Vainshtein, and J. Wheeler, eds., World Scientific 2004, vol. 2, pp. 1173; hep-th/0307041.
- [40] D. Nolland, Phys. Lett. **B584** (2004) 192.
- [41] Ho-Ung Yee, Phys. Lett. **B598** (2004) 139.
- [42] K.A. Samani and M. Zarei, Ann. Phys. (New York) **316** (2005) 466.
- [43] D.E. Diaz and H. Dorn, J. High Energy Phys. **0607** (2006) 022; J. High Energy Phys. **0705** (2007) 046; D.E. Diaz, J. High Energy Phys. **0807** (2008) 103; R. Aros and D.E. Diaz, J. Phys. **A43** (2010) 205402.
- [44] S. De Haro and P. Gao, Phys. Rev. **D76** (2007) 106008.
- [45] G.A. Kerimov, Mod. Phys. Lett. **A22** (2007) 2287.
- [46] H. Gottschalk and H. Thaler, Commun. Math. Phys. **277** (2008) 83.
- [47] S. Moroz, Phys. Rev. **D81** (2010) 066002.
- [48] A. Bayntun, C.P. Burgess, B.P. Dolan and S.-S. Lee, New J.Phys. **13** (2011) 035012.
- [49] D. Kabat, G. Lifschytz and D.A. Lowe, Phys. Rev. **D83** (2011) 106009; D. Kabat, G. Lifschytz, S. Roy and D. Sarkar, Phys. Rev. **D86** (2012) 026004; D. Kabat and G. Lifschytz, Phys. Rev. **D87** (2013) 086004; Phys. Rev. **D89** (2014) 066010.
- [50] X. Bekaert, E. Joung and J. Mourad, Fort. d. Phys. **60** (2012) 882; X. Bekaert and M. Grigoriev, J. Phys. **A46** (2013) 214008.
- [51] X. Gao, M. Kaminski, H. Zeng and H. Zhang, J. High Energy Phys. **1211** (2012) 112.
- [52] S. Ohya, J. High Energy Phys. **1312** (2013) 011.
- [53] S. Hu and T. Li, arXiv:1312.1545 [hep-th].

- [54] R. Hotta, J. Math. Soc. Japan, **23** (1971) 384.
- [55] W. Schmid, Rice Univ. Studies, **56** (1970) 99.
- [56] V.K. Dobrev, G. Mack, V.B. Petkova, S.G. Petrova and I.T. Todorov, *Harmonic Analysis on the  $n$  - Dimensional Lorentz Group and Its Applications to Conformal Quantum Field Theory*, Lecture Notes in Physics, No 63, 280 pages, Springer Verlag, Berlin-Heidelberg-New York, 1977.
- [57] N. Aizawa and V.K. Dobrev, Nucl. Phys. **B828** [PM] (2010) 581.
- [58] S. Ferrara, R. Gatto, A.F. Grillo and G. Parisi, Lett. Nuovo Cim. **4** (1972) 115.
- [59] V.K. Dobrev, J. Phys. **A39** (2006) 5995.
- [60] R.P. Langlands, *On the classification of irreducible representations of real algebraic groups*, Math. Surveys and Monographs, Vol. 31 (AMS, 1988), first as IAS Princeton preprint (1973).
- [61] A.W. Knap and G.J. Zuckerman, in: Lecture Notes in Math. Vol. 587 (Springer, Berlin, 1977) pp. 138; Ann. Math. **116** (1982) 389.
- [62] V.K. Dobrev, Rept. Math. Phys. **25** (1988) 159; first as ICTP Trieste preprint IC/86/393 (1986).
- [63] V.K. Dobrev and P. Moylan, Fort. d. Physik, **42** (1994) 339.
- [64] A. Erdélyi (editor), Bateman Manuscript Project, *Tables of Integral Transforms* Vol.1, McGraw-Hill 1954.
- [65] A.M. Polyakov, JETP Lett. **12** (1970) 381 [Pisma ZhETF **12** (1970) 538].
- [66] N. Aizawa and V.K. Dobrev, in preparation.